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THERMODYNAMICS OF PLASTIC DEFORMATION

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In plastic deformation of materials the relation between the stress tensor  $\sigma_{ik}$  and the deformation tensor  $\epsilon_{ik}$  is not, generally speaking, single valued. However, this relation becomes single valued in the case of the so-called initial charge, when the shearing strain intensity  $\Gamma$  does not diminish with the passing of time. As shown in (1,2), the free energy of a plastically deformed body during the infinitely slow isothermic initial charging  $F_0^{nl}$  can be expressed as

$$F_0^{nl} = F_0^{nl}(\Gamma, \epsilon_{ll}) \quad (1)$$

where  $\Gamma = \sqrt{\frac{2}{3}} \sqrt{\epsilon_{ik}^2 - \frac{1}{3} \epsilon_{ll}^2}$  and  $\epsilon_{ll}$  is the relative change of volume. (1) can be considered as a generalization from experimental data.

Starting from (1), which has been established for the infinitely slow deformation, we can find an expression for the free energy of a body plastically deformed with a final speed  $\dot{\epsilon}_{ik}$ . For this, considering the entities  $\Gamma$  and  $\epsilon_{ll}$  as small, let us develop (1) into a series according to the powers  $\Gamma$  and  $\epsilon_{ll}$  with a precision up to members of the second degree of smallness.

Considering that the uniform compression from all sides does not have any influence on the plastic deformation, we arrive at the expression

$$F_o^{nn} = A + \sqrt{\frac{2}{3}} \sigma_s \frac{\mu - \beta}{\mu} \sqrt{\epsilon_{ik}^2 - \frac{1}{3} \epsilon_{ii}^2} + \beta (\epsilon_{ik}^2 - \frac{1}{3} \epsilon_{ii}^2) + \frac{K}{2} \epsilon_{ii}^2 \quad (2)$$

where  $K$  is the modulus of compression from all sides,  $\mu$  is the shear modulus,  $\sigma_s$  the yield point in elongation,  $\beta$  the flexibility modulus, and  $A$  is the constant.

For the stress tensor we get

$$\sigma_{ik}^{nn} = \frac{\partial F_o^{nn}}{\partial \epsilon_{ik}} = \sqrt{\frac{2}{3}} \sigma_s \frac{\mu - \beta}{\mu} \frac{\epsilon_{ik} - \frac{1}{3} \epsilon_{ii} \delta_{ik}}{\sqrt{\epsilon_{nn}^2 - \frac{1}{3} \epsilon_{nn}^2}} + 2\beta (\epsilon_{ik} - \frac{1}{3} \epsilon_{ii} \delta_{ik}) + K \epsilon_{ii} \delta_{ik} \quad (3)$$

The intensity of shearing stresses

$$S^{nn} = \sqrt{\frac{3}{2}} \sqrt{\sigma_{ik}^2 - \frac{1}{3} \sigma_{ii}^2} = \sigma_s \frac{\mu - \beta}{\mu} + 3\beta \Gamma \quad (4)$$

is the linear intensity function of the shearing stress  $\Gamma$ . This corresponds to the linear strengthening.

For the elastic deformation with the same approximation in the equations (2), (3) and (4) we must assume  $\beta = \mu$ . In the case of an ideal plasticity (absence of hardening)  $\beta = 0$ .

The ratio  $S$  to  $\Gamma$  in actual materials is represented in

Figure 1. The ratio  $S$  to  $\Gamma$  for the approximation examined here is shown in Figure 2.

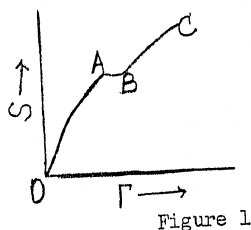


Figure 1

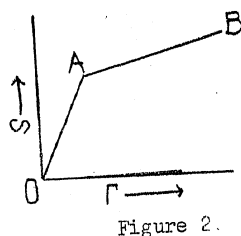


Figure 2.

The segment  $OA$  corresponds to the area of elastic deformation (Hooke's Law), while the segment  $AB$  corresponds to the area of plastic deformation.

At the final speed of plastic deformation (for the initial load) the body will not find itself in a statistic equilibrium and the condition of the body for the isothermic process will be determined by the deformation tensor  $\epsilon_{ik}$  and the relaxation tensor  $\bar{\epsilon}_{ik}$ <sup>(3)</sup>. The relaxation tensor characterizes the degree of removal of the system from the condition of statistic equilibrium.

As the system approaches the condition of statistic equilibrium the tensor  $\epsilon_{ik}$  will tend towards its balance value  $\bar{\epsilon}_{ik}$ . The free energy per volume unit will be the function of the invariants formed with the tensors  $\epsilon_{ik}$  and  $\bar{\epsilon}_{ik}$ . For a homogeneous and isotropic body we can form the following invariants out of the tensors  $\epsilon_{ik}$  and  $\bar{\epsilon}_{ik}$ :

$$I_1^2 = \epsilon_{nn}^2, \quad I_2^2 = \epsilon_{nn}^2 \bar{\epsilon}_{nn}, \quad I_3^2 = \bar{\epsilon}_{nn}^2, \quad I_4^2 = \epsilon_{ik}^2 - \frac{1}{3} \epsilon_{nn}^2,$$

$$I_5^2 = (\epsilon_{ik} - \frac{1}{3} \epsilon_{nn} \delta_{ik})(\bar{\epsilon}_{ik} - \frac{1}{3} \bar{\epsilon}_{nn} \delta_{ik}),$$

$$I_6^2 = \bar{\epsilon}_{ik}^2 - \frac{1}{3} \bar{\epsilon}_{nn}^2.$$

The invariants characterizing the shearing must enter the expression of free energy both in the first and second power; the invariants characterizing the change of volume, only in the second power. The most general invariant in the first power is

$$\sqrt{aI_4^2 + bI_5^2 + cI_6^2}. \text{ Therefore } F^{n\eta} = F^{n\eta}(I_1, I_2, I_3, I_4, I_5, I_6, \sqrt{aI_4^2 + bI_5^2 + cI_6^2})$$

Developing the latter expression into a series and limiting ourselves to members of the second degree of smallness, we get

$$F^{n\eta} = \alpha_1 + \alpha_2 \sqrt{aI_4^2 + bI_5^2 + cI_6^2} + \beta_1 I_1^2 + \beta_2 I_2^2 + \beta_3 I_3^2 + \beta_4 I_4^2 + \beta_5 I_5^2 + \beta_6 I_6^2 \quad (5)$$

Let us introduce the tensor  $\varphi_{ik} = \bar{\epsilon}_{ik} - \bar{\bar{\epsilon}}_{ik}$  where  $\bar{\bar{\epsilon}}_{ik}$  is the value of  $\bar{\epsilon}_{ik}$  determined by the ratio

$$\frac{\partial F^{n\eta}}{\partial \bar{\epsilon}_{ik}} = 0$$

and let us consider that the removal of the system from the condition of statistic equilibrium is insignificant, i. e., that the unbalanced meanings of the invariants  $I_1, I_2, I_3, I_4, I_5, I_6$  differ verily little from their balanced meanings. Therefore (5) can be presented as

$$F^{n\eta} = A + \frac{K}{2} \epsilon_{22}^2 + \beta(\epsilon_{ik}^2 - \frac{1}{3} \epsilon_{22}^2) + \sqrt{\frac{2}{3}} \sigma_s \frac{\mu - \beta}{\mu} \sqrt{\epsilon_{ik}^2 - \frac{1}{3} \epsilon_{22}^2} + N\varphi_{22}^2 + \left(C - \frac{D}{\sqrt{\epsilon_{ik}^2 - \frac{1}{3} \epsilon_{22}^2}}\right)(\varphi_{ik}^2 - \frac{1}{3} \varphi_{22}^2). \quad (6)$$

When  $\varphi_{ik} = 0$  the free energy is minimal. Therefore

$$N > 0, \quad C - \frac{D}{\sqrt{\epsilon_{ik}^2 - \frac{1}{3} \epsilon_{22}^2}} > 0.$$

Because the plastic deformation begins from the entirely determined value of intensity of the shearing strain  $\Gamma_0$ , i.e.,

$$\Gamma = \sqrt{\frac{2}{3}} \sqrt{E_{nm}^2 - \frac{1}{3} E_{nn}^2} \geq \Gamma_0$$

then

$$C > 0, D > 0.$$

In order to transfer (5) into (2) with  $\xi_{ik} = \bar{\xi}_{ik}$  the ratios  $\bar{\xi}_{ik} - \frac{1}{3} \bar{\xi}_{nn} \delta_{ik} = \lambda_1 (\epsilon_{ik} - \frac{1}{3} \epsilon_{nn} \delta_{ik})$ ,  $\bar{\xi}_{nn} = \lambda_2 \epsilon_{nn}$  where  $\lambda_1$  and  $\lambda_2$  are the constants, must be fulfilled.

The relaxation tensor  $\bar{\xi}_{ik}$  satisfies the equations (3):

$$\left| \dot{\phi}_{ik} - \frac{1}{3} \dot{\phi}_{nn} \delta_{ik} = -\tau \left[ \dot{\phi}_{ik} - \frac{1}{3} \dot{\phi}_{nn} \delta_{ik} + \lambda_1 (\dot{\epsilon}_{ik} - \frac{1}{3} \dot{\epsilon}_{nn} \delta_{ik}) \right] \right|_{(7)}$$

$$\left| \dot{\phi}_{nn} = -\tau_2 \left[ \dot{\phi}_{nn} + \lambda_2 \dot{\epsilon}_{nn} \right] \right|_{(8)}$$

where  $\tau$  is the time of strain relaxation in the plastic area,  $\tau_2$  is the time of volume relaxation. In our approximation the time of volume relaxation in the plastic area is equal to the time of volume relaxation in the elastic area.

The solution of equations (7) and (8) gives us

$$\phi_{ik} - \frac{1}{3} \phi_{nn} \delta_{ik} = \psi_{ik} e^{-\frac{t}{\tau}} - \lambda_1 \int_0^t e^{-\frac{t-t'}{\tau}} \left[ \dot{\epsilon}_{ik}(t') - \frac{1}{3} \dot{\epsilon}_{nn}(t') \delta_{ik} \right] dt'_{(9)}$$

$$\varphi_{11} = -\lambda_2 \int_{-\infty}^t e^{\frac{t-t'}{\tau}} \dot{\varepsilon}_{11}(t') dt' \quad (10)$$

where  $\nu_{ik}$  is the value of  $\varphi_{ik} - \frac{1}{3}\varphi_{11}\delta_{ik}$  in the initial moment of time.

The stress tensor in the unbalance state

$$\begin{aligned} \sigma_{ik}^{nn} = \frac{\partial F^{nn}}{\partial \varepsilon_{ik}} = & K \varepsilon_{ik} \delta_{ik} + 2\beta (\varepsilon_{ik} - \frac{1}{3}\varepsilon_{11}\delta_{ik}) + \sqrt{\frac{2}{3}} \frac{\sigma_s \mu - \beta}{\mu} \frac{\varepsilon_{ik} - \frac{1}{3}\varepsilon_{11}\delta_{ik}}{\sqrt{\varepsilon_{nm}^2 - \frac{1}{3}\varepsilon_{nn}^2}} - \\ & - \frac{\eta_2}{\tau_2 \lambda_2} \varphi_{11} \delta_{ik} - \lambda_1 \left( C - \frac{D}{\sqrt{\varepsilon_{nm}^2 - \frac{1}{3}\varepsilon_{nn}^2}} \right) (\varphi_{ik} - \frac{1}{3}\varphi_{11}\delta_{ik}) + \\ & + \frac{D(\varphi_{ss}^2 - \frac{1}{3}\varphi_{ss}^2)}{(\varepsilon_{nm}^2 - \frac{1}{3}\varepsilon_{nn}^2)^{3/2}} (\varepsilon_{ik} - \frac{1}{3}\varepsilon_{11}\delta_{ik}), \end{aligned} \quad (11)$$

where  $\eta_2$  is the second ductility.

With the passage from the elastic area to the plastic the following conditions must be fulfilled:

$$\sigma_{ik}^{yn} = \sigma_{ik}^{nv}, \quad F^{yn} = F^{nv}.$$

From these conditions we determine the constants  $\nu_{ik}$  and the value of  $\Gamma$  in the transition point according to the speed of deformation. This value of the intensity of the shearing strain will be the maximum value of  $\Gamma$  in the area of elastic deformation.

If we say that the evolution of plastic deformation begins

with  $\tau = 0$ , then

$$\lambda \nu_{ik} = - \frac{2}{B-C/\Gamma_0} \frac{\eta_1}{\tau_1} \int_{-\infty}^0 e^{t'/\tau_1} [\dot{\epsilon}_{ik}(t') - \frac{1}{3} \dot{\epsilon}_{ii}(t') \delta_{ik}] dt',$$

$$\Gamma_{MAX} = \Gamma_0 + \frac{4}{\sqrt{6}} \frac{C(\mu-\beta)}{\Gamma_0^2(B-C/\Gamma_0)} \frac{\eta_1^2}{\tau_1^2} \left\{ \int_{-\infty}^0 e^{t'/\tau_1} [\dot{\epsilon}_{ik}(t') - \frac{1}{3} \dot{\epsilon}_{ii}(t') \delta_{ik}] dt' \right\} \quad (12)$$

where  $\Gamma_0 = \sigma_s^0 / 3\mu$ ,  $\tau_1$  is the time of shearing relaxation in the elastic area,  $\eta_1$  is the first ductility and  $\sigma_s^0$  is the elongation yield point in infinitely slow deformation.

For the low deformation speed

$$\begin{aligned} \phi_{ik} &= \frac{1}{3} \phi_{ii} \delta_{ik} = -\lambda_1 \tau (\dot{\epsilon}_{ik} - \frac{1}{3} \dot{\epsilon}_{ii} \delta_{ik}), \\ \phi_{ii} &= -\lambda_2 \tau_2 \dot{\epsilon}_{ii} \end{aligned}$$

and (11) assumes the aspect of

$$\begin{aligned} \sigma_{ik}^{n/l} &= K \epsilon_{ii} \delta_{ik} + 2\beta (\epsilon_{ik} - \frac{1}{3} \epsilon_{ii} \delta_{ik}) + \sqrt{\frac{2}{3}} \sigma_s \frac{\mu-\beta}{\mu} \frac{\epsilon_{ik} - \frac{1}{3} \epsilon_{ii} \delta_{ik}}{\sqrt{\epsilon_{nn}^2 - \frac{1}{3} \epsilon_{nn}^2}} + \\ &+ \eta_2 \epsilon_{ii} \delta_{ik} + \left( C - \frac{D}{\sqrt{\epsilon_{nn}^2 - \frac{1}{3} \epsilon_{nn}^2}} \right) \lambda_1^2 \tau (\epsilon_{ik} - \frac{1}{3} \epsilon_{ii} \delta_{ik}). \quad (13) \end{aligned}$$

Ratios (11), (12), and (13) determine the relation of the yield point and of the maximum elastic deformation to the deformation

speed, the relation of stresses to deformation and to deformation speed in plastic deformation, and also the creep curve, all of which correspond with the experimental data.

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